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THE PROBLEM OF THE EQUILIBRIUM OF A PLATE REINFORCED WITH STIFFENERS†

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The problem of the equilibrium of a non-linear plate reinforced with stiffeners is considered. The idea of a generalized solution of the problem as a critical point of the energy functional of an elastic system is introduced and the existence of a generalized solution of the problem is proved. The convergence of Ritz' method within the framework of this problem and also of the conformal versions of the finite-element method, constructed on the basis of Ritz' method, is validated. Similar problems were discussed in [1–3].

Consider the formulation of the problem of the equilibrium of a loaded plate, reinforced with stiffeners. The plate is described by the following non-linear constitutive relations

$$\begin{aligned} &N_{11}=\frac{Eh}{1-\mu^2}(\varepsilon_{11}+\mu\varepsilon_{22}),\quad N_{22}=\frac{Eh}{1-\mu^2}(\varepsilon_{22}+\mu\varepsilon_{11}),\quad N_{12}=\frac{Eh}{1+\mu}\varepsilon_{12}\\ &M_{11}=D(\varkappa_{11}+\mu\varkappa_{22}),\quad M_{22}=D(\varkappa_{22}+\mu\varkappa_{11}),\quad M_{12}=D(1-\mu)\varkappa_{12}\\ &\varepsilon_{11}=u_{1x}+\frac{1}{2}u_{3x}^2,\quad \varepsilon_{22}=u_{2y}+\frac{1}{2}u_{3y}^2,\quad \varepsilon_{12}=u_{1y}+u_{2x}+u_{3x}u_{3y}\\ &\varkappa_{11}=-u_{3xx},\quad \varkappa_{22}=-u_{3yy},\quad \varkappa_{12}=-u_{3xy} \end{aligned} \tag{1}$$

where E, μ and D are the elasticity constants, while the subscripts x and y denote differentiation with respect to the corresponding variable. The equilibrium equations of the plate have the form

$$\begin{aligned} &D\nabla^4u_3-N_{11}u_{3xx}-N_{22}u_{3yy}-2N_{12}u_{3xy}-F_3=0\\ &\nabla^2u_1+\frac{1+\mu}{1-\mu}(u_{1y}+u_{2x})_x+\frac{2}{1-\mu}(u_{3x}u_{3xx}+\mu u_{3y}u_{3xy})+u_{3y}u_{3xy}+u_{3x}u_{3yy}+F_1=0\\ &\nabla^2u_2+\frac{1+\mu}{1-\mu}(u_{1y}+u_{2x})_y+\frac{2}{1-\mu}(u_{3y}u_{3yy}+\mu u_{3x}u_{3xy})+u_{3x}u_{3xy}+u_{3y}u_{3xx}+F_2=0 \end{aligned}$$

where F_i are the components of the external load vector.

We will assume that the closed region Ω, occupied by the plate, is bounded and has a piecewise-smooth boundary (all the angles at the corner points of the boundary are non-zero). The plate is reinforced with a finite number n of rectilinear stiffeners. The stiffeners are described using the classical theory of rods.

Depending on the ratio of the stiffness of the stiffeners to that of the plate itself we can consider different versions of the equations describing the stiffeners. If the stiffness of the stiffeners is comparatively high, it make sense to use linear equations to describe these stiffeners. If the stiffeners are not too stiff, we must take non-linear relations. The side flexure of the stiffeners can often be neglected, but this is not done here.

We will consider the linear rod version of the model of a stiffener. To describe the rod we choose a rectangular system of coordinates (s, ξ^1, ξ^2), where the coordinate lines (ξ^1, ξ^2) pass along the principal central axes of the rod section, and s is a natural parameter of the length of the neutral axis of the rod.

The displacement of a point on the neutral axis of the rod in projections onto the coordinate lines (s, ξ^1, ξ^2) is denoted by (u, w_1, w_2). The main characteristics of the deformation of the stiffener are specified as follows. The angle of rotation of the transverse cross-section of the rod around the ξ^i axis

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is $\varphi_i = dw_i/ds$. Suppose φ is the angle of rotation of the transverse cross-section around the s axis and the angle of relative torsion of the cross-section is $\chi = d\varphi/ds$. We will also introduce the axial strain $\varepsilon = du/ds$ and the curvatures $\varkappa_i = -d\varphi_i/ds = -d^2w_i/ds^2$ $(i = 1, 2)$. These characteristics of the deformation are related to the longitudinal force N, the torsional moment M and the bending moments M_1 and M_2 by the relations

$$T = B\varepsilon, \quad M = D_x\chi, \quad M_1 = D_1\varkappa_1, \quad M_2 = D_2\varkappa_2$$

where B, D_χ, D_1, D_2 are the stiffnesses of the rod, expressed in terms of the physical constants of the material and the geometrical parameters of the rod. The equilibrium equations of such a rod are well known and will not be given here (see, for example, [4]).

For a complete formulation of the problem we must state the conditions for the deformation of the plate and of the stiffeners to be compatible. These equations will not be specified in detail here. It should be noted that the set of quantities (u, w_1, w_2, φ) for the rod is expressed in terms of the displacement vector of the corresponding point of the plate (u_1, u_2, u_3) and its derivatives by the linear relations (see [4])

$$u = u(u_1,u_2,u_3), \quad w_1 = w_1(u_1,u_2,u_3), \quad w_2 = w_2(u_1,u_2,u_3)$$
$$\varphi = \varphi(u_1,u_2,u_3) \tag{2}$$

Henceforth all the deformation characteristics of the stiffener are assumed to be expressed in terms of the corresponding components of the displacement vector of the plate by relations (2).

Each of the n stiffeners, denoted by $R\nu$ $(\nu = 1, \ldots, n)$, may have different physical characteristics. The subscript ν will be dropped in the notation for the parameters of the νth stiffener.

To formulate the equilibrium problem we need to specify the boundary conditions. It is required that, along the normal to the middle plane of the direction, the plate is clamped at three points (x_i, y_i) which do not lie on a single straight line

$$u_3(x_i, y_i) = 0, \quad i = 1, 2, 3 \tag{3}$$

We can also assume that there is a part of the boundary of the region Γ_1 where the following condition is satisfied

$$u_3 |_{\Gamma_1} = 0 \tag{4}$$

and a part Γ_2 where the following condition is satisfied

$$\partial u_3 / \partial n |_{\Gamma_2} = 0 \tag{5}$$

The set of functions belonging to $C^{(4)}(\Omega)$ and which satisfy conditions (3)–(5) will be denoted by C_4.

For the tangential displacements $\mathbf{v} = (u_1, u_2)$, we shall require that boundary conditions are satisfied such that the Korn inequality holds for the planar theory of elasticity

$$\int_\Omega (u_1^2 + u_2^2 + u_{1x}^2 + u_{1y}^2 + u_{2x}^2 + u_{2y}^2)dxdy \le m \int_\Omega \{u_{1x}^2 + (u_{1y} + u_{2x})^2 + u_{2x}^2\}dxdy$$

One of the possible versions where this inequality is satisfied with constant m, which is independent of $\mathbf{v}$, is the condition

$$u_1 |_{\Gamma_3} = 0, \quad u_2 |_{\Gamma_3} = 0 \tag{6}$$

where Γ_3 is a certain part of the boundary contour $\partial\Omega$ of the region.

We will introduce the set C_2 of vector functions $\mathbf{v} = (u_1, u_2)$, each component of which belongs to the space $C^{(2)}(\Omega)$ and such that condition (6) is satisfied.

We can also assume that other parts of the boundary of the region are rigidly clamped (then the corresponding conditions must be included in the definition of the sets C_4 or C_2), or there is elastic support (the corresponding elastic support energy must be included in the total energy functional), or there are specified external loads. In the last case, the energy functional includes an integral term (the integral along the boundary of the region), equal to the work of the external forces on the boundary. These conditions will not be written down in differential form, since they are well known and can be obtained by standard methods from the variational formulation of the problem. To fix our ideas we will assume that conditions (3)–(6) are satisfied.

Consider a plate under a normal load ($F_1 = F_2 = 0$). As we know from the general theory, the solution of the problem of the deformation of a plate reinforced with stiffeners, on a set of vector functions satisfying the geometrical conditions of the clamping of the edge, is obtained as a vector function **u**, satisfying the plate edge clamping conditions and minimizing the total energy functional, in which all the terms are expressed in terms of the displacement vector of the middle plane of the plate

$$I(\mathbf{u}) = \| u_3 \|_4^2 + \frac{1}{2}\int_\Omega (N_{11}\varepsilon_{11} + N_{22}\varepsilon_{22} + N_{12}\varepsilon_{12})dxdy + \Sigma - \int_\Omega F_3u_3dxdy - \int_{\partial\Omega} f_3u_3ds \tag{7}$$

$$\| u_3 \|_4^2 = \frac{1}{2}\int_\Omega (M_{11}\varkappa_{11} + M_{22}\varkappa_{22} + M_{12}\varkappa_{12})d\Omega$$

$$\Sigma = \sum_{\nu=1}^{n} \frac{1}{2}\int_{R_\nu} (T\varepsilon + M\chi + M_1\varkappa_1 + M_2\varkappa_2)ds$$

where f_3 is the normal load specified on the edge of the plate. The last two integrals on the right-hand side of the first equation of (7) describe the work of the external forces in the displacement u_3. The expressions under the summation sign describe the contribution to the potential energy function $I_1(\mathbf{u})$ for a plate without stiffeners solely in the quantity Σ. It should be noted that the integrated in the integrals under the summation sign is a positive-definite quadratic form of the variables ε, χ, $\varkappa_1$, $\varkappa_2$ (here it is assumed that the displacement variables of points of the rod are expressed in terms of the coordinates of the plate displacement vector by means of formulae (2)).

The complete equations of equilibrium in displacements for a plate with stiffeners and the natural boundary conditions are obtained in the usual way for the variational technique by processing the equation $\delta I(\mathbf{u}) = 0$, where $\delta I(\mathbf{u})$ is the first variation of the functional $I(\mathbf{u})$.

For the classical formulation of this problem the conditions formulated along the reinforcing lines of the stiffeners in the Ω plane, become part of the boundary conditions. They are natural boundary conditions which describe the common deformation of the stiffener and the plate. A curious feature of these "boundary" conditions is the fact that the highest order of the differentiation in the boundary conditions is four, which is equal to the order of the highest derivative of u_3. This non-classical combination of the order of the derivatives in the equations and boundary conditions nevertheless does not imply that the mathematical scheme used to investigate the general solvability of the problem will be troublesome.

For the mathematical formulation of the problem, we will introduce a functional energy space in which the solution will be found. The energy norm in this space includes all positive quadratic terms of the complete energy functional. We will first introduce auxiliary functional spaces. The space E_4 is the closure of the functions $u(x, y) \in C_4$ in the energy norm $\|\cdot\|_4$, which is associated with the scalar product

$$(u,\upsilon)_4 = \frac{1}{2}\int_\Omega [M_{11}(u)\varkappa_{11}(\upsilon) + M_{22}(u)\varkappa_{22}(\upsilon) + M_{12}(u)\varkappa_{12}(\upsilon)]d\Omega, \quad \| u \|_4 = (u,u)_4^{1/2}$$

The space E_2 is the closure of the vector functions $\mathbf{v}(x, y) \in C_2$ in the energy norm $\|\cdot\|_2$, induced by the scalar product

$$(\mathbf{v}_1,\mathbf{v}_2)_2 = \frac{1}{2}\int_\Omega [N_{e11}(\mathbf{v}_1)e_{11}(\mathbf{v}_2) + N_{e22}(\mathbf{v}_1)e_{22}(\mathbf{v}_2) + N_{e12}(\mathbf{v}_1)e_{12}(\mathbf{v}_2)]d\Omega$$

where

$$N_{e11} = \frac{Eh}{1-\mu^2}(e_{11} + \mu e_{22}), \quad N_{e22} = \frac{Eh}{1-\mu^2}(e_{22} + \mu e_{11}), \quad N_{e12} = \frac{Eh}{1+\mu}e_{12}$$

$$e_{11} = u_{1x}, \quad e_{22} = u_{2y}, \quad e_{12} = u_{1y} + u_{2x}$$

For the variables $M_{11}, \ldots, \varepsilon_{12}$ in parentheses we indicate the arguments u, υ or $\mathbf{v}$, which must be substituted into relations (1).

We introduce two norms into the set $C_2 \times C_4$ by the equations

$$\| \mathbf{u} \|_0^2 = \| \mathbf{v} \|_2^2 + \| u_3 \|_4^2, \quad \mathbf{v} = (u_1,u_2); \quad \| \mathbf{u} \|_1^2 = \| \mathbf{u} \|_0^2 + \Sigma$$

The scalar products corresponding to these norms are obtained by a method that is standard for Hilbert spaces. The closures of the set $C_2 \times C_4$ in these norms are Hilbert spaces, and they are denoted by E_0 and E_1, respectively.

It is obvious that the elements of the space E_1 will be elements of the space E_0. We know [1, 2] that the space E_0 is a subspace of the Cartesian product of the Sobolev spaces $W = W_1^{(1)}(\Omega) \times W_1^{(1)}(\Omega) \times W_2^{(2)}(\Omega)$ [5], the norm of which is equivalent to the norm $\|\cdot\|_0$ introduced above. Further, we use the corresponding Sobolev imbedding theorems [5] in the space W (and, consequently, also in E_0). The space E_1 has, in a certain sense, "smoother" elements than the space E_0, since, on sections of the straight lines R_v the vector functions from E_1 have a greater degree of smoothness than follows from the theorems of imbedding in E_0. It should be noted that the component $u_3(x, y)$ of an arbitrary element from E_0, by virtue of Sobolev's imbedding theorem, is a continuous function, so the clamping condition (3) therefore also has meaning in the spaces E_0 and E_1.

Definition. The stationary point of the energy functional $I(\mathbf{u})$ in the space E_1 is called the generalized solution $\mathbf{u}$ of the plate equilibrium problem.

Thus, the generalized solution satisfies the equation $\delta I(\mathbf{u}) = 0$. As we know, this equation is the basis of the solution of this problem by the Bubnov–Galerkin method and, consequently, by the finite-element method also.

To correct the generalized formulation of the problem we need to impose limitations on the external loads. The terms of the functional $I(\mathbf{u})$ corresponding to the work of the external forces must have meaning for any vector functions $\mathbf{u} \in E_1$. The class of such loads F_3 and f_3 will be denoted by E^*. By virtue of the imbedding theorems, the sufficient conditions for a load to belong to the class E^* are $F_3 \in L(\Omega), f_3 \in L(\partial\Omega)$.

The energy functional $I(\mathbf{u})$ has the same structure as the energy functional for the plate

$$I(\mathbf{u}) = \| \mathbf{u} \|_1^2 + \Psi(\mathbf{u})$$

where $\Psi(\mathbf{u})$ is a weakly continuous functional in the space E_1. Moreover, the functional $\Psi(\mathbf{u})$ is identical with the corresponding functional for the plate It follows from the general theory (see also [1, 2]) that to prove the generalized solvability of the problem it is sufficient to show that the functional $I(\mathbf{u})$ is an increasing functional, i.e. $I(\mathbf{u}) \to \infty$ if $\| \mathbf{u} \|_1 \to \infty$.

Lemma. Suppose the load belongs to the class E^*. Then functional $I(\mathbf{u})$ is an increasing functional.

This is proved using the same scheme as for the plate. Namely, we show that it follows from the inequality $I(\mathbf{u}) \leqslant m$ that a constant $M < \infty$ exists such that $\| \mathbf{u} \|_1 \leqslant M$.

Thus, suppose $I(\mathbf{u}) \leqslant m$. Since the inequality

$$\left| \int_\Omega F_3 u_3 dx dy + \int_{\partial\Omega} f_3 u_3 ds \right| \leq c \| u_3 \|_4$$

is satisfied with a certain constant c, which is independent of u_3, it follows directly from the inequality $I(\mathbf{u}) \leqslant m$ that $\| u_3 \|_4 \leqslant m_1$. However, it follows directly from the inequality $I(\mathbf{u}) \leqslant m$ that $\Sigma \leqslant m_2$, and then also that $\| \mathbf{v} \|_2 \leqslant m_3$, $\mathbf{v} = (u_1, u_2)$, where m_i are certain constants.

The following theorem follows directly from the above lemma and from the general theory and the method described previously in [1–3].

Theorem. Suppose the plate is acted upon by a normal load of class E^*. In this case:

1. there is at least one generalized solution of the problem of the equilibrium of a shell belonging to the space E_4, which makes the energy functional $I(\mathbf{u})$ a minimum;
2. any minimizing functional $I(\mathbf{u})$ of the sequence $\mathbf{u}_n$ contains a subsequence, which converges strongly in the space E_4 to the generalized solution of the problem;
3. the system of equations of the approximate solution of the problem by Ritz' method and (thereby by the Bubnov–Galerkin method and, consequently, by any conformal version of the finite-element method) is solvable at each step and contains a subsequence which strongly converges to the generalized solution of the problem in E_4; moreover, any weakly converging subsequence of the approximations converges strongly to a certain generalized solution of the problem.

The solution of this non-linear problem is, in general, non-unique.

To investigate this problem we will use a topological approach, employed previously in [1–3] in the theory of shallow shells. Using this approach, in the condition of the problem, we can introduce loads which act parallel to the middle plane of the plate. Here, during the course of the proofs, additional limitations on the method of clamping the edge of the plate in directions parallel to the middle plane arise (for more detail see [2]).

Finally, we note that, when solving the problem by the finite-element method, an algebraic system of equations is obtained in which the equations containing the junction points on the stiffeners have a form which differs from the form of the equations for the junction points outside the stiffeners, but the structure of the equations themselves remains essentially unchanged.

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